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DYNAMIC THRESHOLDS OF GEOMETRIC CONSISTENCY INDEX ASSOCIATED WITH PAIRWISE COMPARISON MATRIX

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Abstract. Pairwise comparison matrix (PCM) has been widely employed in the multi-criteria decision-making (MCDM) problems to rank the criteria and alternatives according to the considered criteria in Analytic Hierarchy Process (AHP). The PCM should have the acceptable consistency before deriving a priority vector from it. Approximate thresholds of geometric consistency index (GCI) and consistency ratio (CR) have been proposed to test whether the PCM has the acceptable consistency. However, approximate thresholds of GCI and CR always suffer from some criticisms and disagreements in existing literature. In this paper, we try to induce dynamic thresholds of GCI by combining hypothesis testing and random index (RI), which vary with the order of the PCM, significance level and assessment level of decision maker. The induced dynamic thresholds of GCI may explain different (or conflicting) results obtained by approximate thresholds of GCI and CR and avoid the unnecessary revisions of some judgments of the PCM for the desired consistency. Finally, several numerical examples and real-world decision-making problems are examined and compared with existing decision-making methods to illustrate the performance of dynamic thresholds of GCI.

Keywords: analytic hierarchy process, pairwise comparison matrix, geometric consistency index, dynamic thresholds.

JEL Classification: C13, C39.

Introduction

Multi-criteria decision-making (MCDM) (Brugha, 2004) has been widely applied in various fields to solve real-world decision-making problems (Jin et al., 2021). Analytic Hierarchy Process (AHP) developed by Saaty (1977), as one of the most popular MCDM methods, has

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been extensively applied to evaluate the sustainability indicators selection process and select the best alternative considering economic, social, and environmental criteria (Georgiou et al., 2015; Hamchaoui et al., 2015; Marques et al., 2015). AHP is converted to the multiplicative structure that is referred to as multiplicative AHP (Barzilai et al., 1987). Brugha (1996) suggested that it is inappropriate to use the AHP to synthesize qualitatively identical attributes of an alternative. Pairwise comparison matrix (PCM) in AHP has been widely employed in the MCDM problems to rank the criteria and alternatives according to the considered criteria (Barzilai & Golany, 1994; Ishizaka & Labib, 2011; Saaty, 2013; Kou et al., 2014, 2016, 2020; Csató & Petróczy, 2020). Generally, the PCM is not consistent due to the complexity of decision-making problem and the cognitive deficiency of decision maker. The lack of acceptable consistency easily leads to inconsistent conclusions (Jin et al., 2020). Given that the consistency is unattainable in practice, a degree of the inconsistency is acceptable (Amenta et al., 2020). That is to say, the PCM should have the acceptable consistency before deriving the priority vector derived from it. As highlighted by Aguarón et al. (2021), both the judgments and the derived priority vector will be close to the initial values. Therefore, it is necessary to test whether the PCM has the acceptable consistency.

Consistency analysis and consistency improvement are two important issues for various preference relations (Lin et al., 2014; Jin et al., 2016, 2020). Recently, more than ten consistency indices (also called inconsistency indices) have been proposed to measure the consistency level of the PCM constructed on 1-9 scale, such as consistency ratio (CR) (Saaty, 1977), geometric consistency index (GCI) (Crawford & Williams, 1985), consistency measure (CM) (Salo & Hamalainen, 1997; Amenta et al., 2018), harmonic consistency index (HCI) (Stein & Mizzi, 2007), cosine consistency index (CCI) (Kou & Lin, 2014), and other consistency indices (Brunelli, 2018). Many researchers have conducted in-depth analysis on consistency indices (Koczkodaj, 1993; Duszak & Koczkodaj, 1994; Peláez & Lamata, 2003; Gass & Rapcsák, 2004; Fedrizzi & Giove, 2007; Bozóki & Rapcsák, 2008; Cavallo & D'Apuzzo, 2009, 2010; Kułakowski, 2015; Grzybowski, 2016; Fedrizzi & Ferrari, 2018; Dixit, 2018). For example, Cavallo (2020) looked for functional relations and correlations among nine inconsistency indices. Some authors have applied an axiomatic approach by reasonable properties required from an inconsistency index (Brunelli & Fedrizzi, 2015; Brunelli, 2016, 2017; Koczkodaj & Szwarc, 2014; Koczkodaj & Urban, 2018; Csató, 2018a, 2019a).

As the maximum level permitted for accepting the inconsistency of the PCM, the threshold of consistency index should be provided beforehand for the consistency testing of the PCM. If the value of consistency index is less than the corresponding threshold, then the PCM has the acceptable consistency; Otherwise, the PCM would be modified for the acceptable consistency. In fact, many consistency indices lack extensive applications because of the absence of thresholds associated with them. As suggested by Monsuur (1997), the threshold of consistency index is the useful tool for decision maker to move towards his (her) true preferences.

In existing literature, several thresholds of CR and GCI for the PCM constructed on 1-9 scale have been proposed. More specifically, the first threshold (0.10) of CR was proposed by Saaty (1980), and then was revised by Saaty (1994). That is, 0.05 for $n = 3$, 0.08 for $n = 4$, and 0.10 for $n > 4$. Approximate thresholds of GCI were induced by Aguarón and Moreno-Jiménez (2003). That is, 0.3147 for $n = 3$, 0.3526 for $n = 4$, and 0.370 for $n > 4$. Moreover, the transitivity thresholds of GCI were computed by simulation (Amenta et al., 2020). Aguarón and Moreno-Jiménez (2003) indicated that approximate thresholds of GCI have an interpretation analogous to the threshold (0.10) of CR. Thus, the above-mentioned approximate thresholds of GCI will be confronted with the same criticisms and disagreements as the threshold (0.10) of CR in dealing with complex decision-making problems, which are listed below:

1) The threshold (0.10) of CR may allow the contradictory judgments in the PCMs or reject reasonable PCMs (Karapetrovic & Rosenbloom, 1999; Kwiesielewicz & Uden, 2004; Banae & Vansnick, 2008). As highlighted by Siraj et al. (2012), the threshold (0.10) of CR may accept many intransitive PCMs or reject many transitive ones.
2) The threshold (0.10) of CR is typically relaxed for lower-order PCM and is stricter for higher-order PCM (Lin et al., 2013, 2014). As highlighted by Bozóki et al. (2013), the order of the matrix has impact on the inconsistency of the PCM.
3) It is ill-suited that approximate thresholds of CR and GCI are fixed ($CR' = 0.10$, $GCI' = 0.370$) for more than four-order PCM. As highlighted by Siraj et al. (2015), a value of CR lower than 0.1, representing an acceptable consistency, does not ensure the transitivity of comparisons.

There are two most popular prioritization procedures in multiplicative AHP: eigenvector method (EM) (Saaty, 1977) and logarithmic least squares method (LLSM) (Crawford & Williams, 1985) (also called row geometric mean method (RGMM)). Barzilai et al. (1987) argued that LLSM over performs EM in multiplicative AHP. The row geometric mean vector presented by Crawford and Williams (1985) in the basis of statistical and logarithmic least squares considerations has been supported by some researchers (Barzilai et al., 1987; Barzilai & Golany, 1994). Subsequently, some researchers further investigated GCI associated with LLSM from various scientific perspectives (Barzilai, 1997; Altuzarra et al., 2007; Lundy et al., 2017; Csató, 2018b, 2019b; Amenta et al., 2020). However, many questions remain about the choice of the right cut-off rule to declare the inconsistency of a PCM (Amenta et al., 2020). Until now, there few reports on dynamic thresholds of GCI for the consistency testing of the PCM. In order to overcome these limitations, the main aims of this study are listed below:

1) To induce dynamic thresholds of GCI varying with the order of the PCM, significance level and assessment level of decision maker (details given in Table 3 and Example 1).
2) To explain the conflicting results obtained by different consistency thresholds with the help of significance level for the consistency testing of the PCM (details given in Examples 1 and 3).
3) To avoid the unnecessary revisions of some judgments of the PCM for only improving the value of consistency index (details given in Examples 2, 3 and 4).

According to these aims, we next try to induce dynamic thresholds of GCI by combing hypothesis testing and random index (RI).

The rest of this paper is structured as follows. Section 1 briefly reviews the related research: definitions, theorems, prioritization procedures and consistency indices. Section 2 induces dynamic thresholds of GCI by combing hypothesis testing and RI. Section 3 illustrates the performance of dynamic thresholds of GCI using numerical examples and real-world decision-making problems. The last Section concludes with some comments about dynamic thresholds of GCI.

1. Related research

1.1. Definitions and theorems

Several definitions and theorems related to dynamic thresholds of GCI are briefly introduced as follows:

Definition 1 (Saaty, 1980). Matrix $A=(a_{ij})_{n\times n}$ is positive reciprocal matrix if $a_{ij}>0$, $a_{ii}=1$ and $a_{ij}=1/a_{ji}$ for all $i,j\in\{1,2,\cdots,n\}$.

Definition 2 (Saaty, 1980). Positive reciprocal matrix $A=(a_{ij})_{n\times n}$ is consistent if $a_{ij}=a_{il}a_{lj}$ for all $i, j, l\in\{1,2,\cdots,n\}$.

Definition 3 (Vargas, 1982). Matrix $A=(a_{ij})_{n\times n}$ is random reciprocal matrix if a_{ij} are random variables that satisfy the property that $a_{ij}>0$, $a_{ii}=1$ and $a_{ij}=1/a_{ji}$ for all $i,j\in\{1,2,\cdots,n\}$.

Definition 4 (Escobar & Moreno-Jiménez, 2000). Random variable X defined in R^+ is reciprocal if X and $\frac{1}{X}$ are identically distributed. That is to say, $P(X\le x)=P\left(\frac{1}{X}\le x\right)\forall x\in R^+$, where $P(\cdot)$ is the probability function.

Definition 5 (Bernardo & Smith, 1994). If random variable X ($X > 0$) follows lognormal distribution $LN(\mu,\sigma^2)$, then $\ln(X)$ follows normal distribution $N(\mu,\sigma^2)$.

Definition 6 (Devore, 2000). If $X_1,X_2,\cdots,X_n$ are independent random variables and $X_i\sim N(0,1)(i=1,2,\cdots,n)$, then $\chi^2=X_1^2+\cdots+X_n^2\sim\chi^2(n)$ and $E(\chi^2(n))=n$.

Theorem 1 (Escobar & Moreno-Jiménez, 2000). If $A=(a_{ij})_{n\times n}$ is a random reciprocal matrix, and the elements $a_{ij}(1\le i<j\le n)$ are independent and reciprocal random variables, then the priorities $v_i\,(i=1,2,\cdots,n)$ derived by LLSM are reciprocal random variables.

Theorem 2 (Aguarón & Moreno-Jiménez, 2003). Given that the priorities $v_i\,(i=1,2,\cdots,n)$ are derived from the PCM $A=(a_{ij})_{n\times n}$ by LLSM, then $GCI=\frac{2n}{n-2}CI+o(\varepsilon^3)$, where $CI=(\lambda-n)/(n-1)$ (λ is the principal eigenvalue of $A=(a_{ij})_{n\times n}$) and $\varepsilon=Max\{|\ln e_{ij}|\}$ $(e_{ij}=v_j a_{ij}/v_i)$.

1.2. Prioritization procedures and consistency indices

It is well known that CR and GCI are the corresponding consistency indices for EM and LLSM, respectively. LLSM, EM, GCI and CR are briefly introduced as follows:

Logarithmic least squares method (LLSM)

The desired priority vector $V=(v_1,v_2,\cdots,v_n)^T$ is a solution of the constrained optimization problem:

$$\text{Min}\sum_{i=1}^{n}\sum_{j=1}^{n}(\ln a_{ij}-\ln v_i+\ln v_j)^2 \text{ subject to } v_i>0\,(i=1,2,\cdots,n),$$

where $A=(a_{ij})_{n\times n_1}$ is the PCM. The priorities which minimize the above function are given by $v_i=(\prod_{j=1}^{n}a_{ij})^{\frac{1}{n}}\left(i=1,2,\cdots,n\right)$ (without the normalization factor).

Eigenvector method (EM)

The desired priority vector $W=(w_1,w_2,\cdots,w_n)^T$ is derived by solving the linear system: $AW=\lambda W$ subject to $\sum_{i=1}^{n}w_i=1,\ w_i>0\left(i=1,2,\cdots,n\right)$, where $A=(a_{ij})_{n\times n}$ is the PCM, and λ is the principal eigenvalue of $A=(a_{ij})_{n\times n}$.

Geometric consistency index (GCI)

GCI is defined by

$$GCI=\frac{2}{\left(n-1\right)\left(n-2\right)}\sum_{1\le i<j\le n}\left[\ln(a_{ij})-\ln(\frac{v_i}{v_j})\right]^2, \tag{1}$$

where $A=(a_{ij})_{n\times n}$ is the PCM, and $V=(v_1,v_2,\cdots,v_n)^T$ is the priority vector derived from $A=(a_{ij})_{n\times n}$ by LLSM.

Consistency ratio (CR)

CR is defined by

$$\mathrm{CR}=\frac{CI}{RI}, \tag{2}$$

where *RI* is the random index under the condition of 1–9 scale, and *CI* is consistency index that is denoted by $CI=\left(\lambda-n\right)/\left(n-1\right)$, where λ and n are the principal eigenvalue and the order of the PCM, respectively.

Generally, GCI(CR) should be as small as possible since $GCI=0$ $(CR=0)$ if and only if the PCM is consistent. The PCM should be revised for the acceptable consistency when GCI(CR) is greater than the corresponding threshold.

As mentioned above, approximate thresholds of CR and GCI for different order PCMs are shown in Table 1 (Saaty, 1994; Aguarón & Moreno-Jiménez, 2003).

Table 1. Approximate thresholds of CR and GCI for different order PCMs

	$n=3$	$n=4$	$n>4$
CR'	0.05	0.08	0.10
GCI'	0.3147	0.3526	0.370

2. Dynamic thresholds of GCI

Generally, complex scenario leads to uncertain contexts in which the preferences of decision-makers eliciting their knowledge are viewed as random variables. From the statistical point of view, the rejection of a reasonable PCM would be type I error; allowing the contradictory judgments in the PCM would be type II error. It is well known that type I error depends on the significance level. A suitable significance level may be selected according to the actual situation. Thus, the thresholds of GCI are related to the selected significance level. In order to test whether different order PCMs have the accepted consistency, we try to induce dynamic thresholds of GCI by combining hypothesis testing and RI.

Let the PCM $A=(a_{ij})_{n\times n}$ be a positive reciprocal matrix constructed on 1–9 scale, and $V=(v_1,v_2,\cdots,v_n)^T$ be the priority vector derived from $A=(a_{ij})_{n\times n}$ by certain prioritization procedure. When $A=(a_{ij})_{n\times n}$ is consistent, it holds that

$$a_{ij}=v_i/v_j \quad (i,j\in\{1,2,\cdots,n\}). \tag{3}$$

In view of this, the ratios v_i/v_j are approximate values of a_{ij} for all $i,j\in\{1,2,\cdots,n\}$ when $A=(a_{ij})_{n\times n}$ is not consistent. Thus, the error terms are denoted by

$$e_{ij}=v_j a_{ij}/v_i \quad (i,j\in\{1,2,\cdots,n\}). \tag{4}$$

Taking the logarithm for both sides of (4) and letting $\varepsilon_{ij}=\ln(e_{ij})$, an equivalent expression is

$$\varepsilon_{ij}=\ln a_{ij}+\ln v_j-\ln v_i \quad (i,j\in\{1,2,\cdots,n\}). \tag{5}$$

Since $A=(a_{ij})_{n\times n}$ is a positive reciprocal matrix, it follows that $a_{ij}>0$, $a_{ij}=1/a_{ji}$ and $a_{ii}=1$ for all $i,j\in\{1,2,\cdots,n\}$ according to Definition 1. Accordingly, it follows that $e_{ij}>0$, $e_{ij}=1/e_{ji}$ and $e_{ii}=1$ for all $i,j\in\{1,2,\cdots,n\}$ from (4). It further follows that $\varepsilon_{ij}=-\varepsilon_{ji}$ and $\varepsilon_{ii}=0$ for all $i,j\in\{1,2,\cdots,n\}$ from (5).

Escobar and Moreno-Jiménez (2000) suggested that the judgments in the PCM are reciprocal random variables and that lognormal distribution is reciprocal. Thus, we may assume that $a_{ij}(1\le i<j\le n)$ in $A=(a_{ij})_{n\times n}$ are independent random variables following lognormal distribution. Moreover, we further assume that the priorities $v_i\left(1=1,2,\cdots,n\right)$ in $V=(v_1,v_2,\cdots,v_n)^T$ are derived from $A=(a_{ij})_{n\times n}$ by LLSM. According to Theorem 1, the priorities $v_i\left(1=1,2,\cdots,n\right)$ are independent random variables following lognormal distribution. It follows that $\ln(a_{ij})(1\le i<j\le n)$ and $\ln v_i\left(1=1,2,\cdots,n\right)$ follow normal distribution according to Theorem 2. Moreover, $\ln(a_{ij})(1\le i<j\le n)$ and $\ln v_i\left(1=1,2,\cdots,n\right)$ are independent random variables as $a_{ij}(1\le i<j\le n)$ and $v_i\left(1=1,2,\cdots,n\right)$ are independent random variables. It is concluded that $\varepsilon_{ij}=\ln(a_{ij})-\ln(v_i)+\ln(v_j)(1\le i<j\le n)$ follow normal distribution. We further assume that $\varepsilon_{ij}(1\le i<j\le n)$ are independent and follow the normal distribution with mean 0 and variance σ^2. That is

$$\varepsilon_{ij}\sim N\left(0,\sigma^2\right)(1\le i<j\le n). \tag{6}$$

It follows that $\frac{\varepsilon_{ij}}{\sigma}\sim N\left(0,1\right)(1\le i<j\le n)$ by standardization strategy. Thus, the test statistic is denoted by

$$\chi^2=\sum\nolimits_{1\le i<j\le n}\frac{\varepsilon_{ij}^2}{\sigma^2}. \tag{7}$$

According to Definition 6, the test statistic χ^2 follows Chi-square distribution with the degree of freedom $n\left(n-1\right)/2$. That is

$$\sum\nolimits_{1\le i<j\le n}\frac{\varepsilon_{ij}^2}{\sigma^2}\sim\chi^2\left(n\left(n-1\right)/2\right). \tag{8}$$

If the PCM $A=(a_{ij})_{n\times n}$ is consistent, then $\chi^2=0$ since $\varepsilon_{ij}=\ln e_{ij}=0$ $(e_{ij}=1)$ for all $i,j\in\{1,2,\cdots,n\}$. If the PCM $A=(a_{ij})_{n\times n}$ is not consistent, then $\chi^2>0$. Thus, the consistency testing of the PCM $A=(a_{ij})_{n\times n}$ are equivalently described as follows:

Given the significance level α, if $0<\chi^2<\chi^2_\alpha$, then the PCM $A=(a_{ij})_{n\times n}$ has the acceptable consistency. If $\chi^2 \ge \chi^2_\alpha$, then the PCM $A=(a_{ij})_{n\times n}$ has not the acceptable consistency.

From (1) and (5), we obtain

$$GCI=\frac{2}{(n-1)(n-2)}\sum\nolimits_{1\le i<j\le n}\varepsilon_{ij}^2. \tag{9}$$

From (7) and (9), we obtain

$$\chi^2=\sum\nolimits_{1\le i<j\le n}\frac{\varepsilon_{ij}^2}{\sigma^2}=\frac{1}{2\sigma^2}(n-1)(n-2)GCI. \tag{10}$$

That is,

$$GCI=\frac{2\sigma^2\chi^2}{(n-1)(n-2)}. \tag{11}$$

From (11), it follows that $GCI=0$ when $A=(a_{ij})_{n\times n}$ is consistent, and that $GCI>0$ when $A=(a_{ij})_{n\times n}$ is not consistent. Accordingly, the consistency testing of the PCM ($A=(a_{ij})_{n\times n}$) is equivalently described as follows:

Given the significance level α, if $0<GCI<GCI'_\alpha(n)$, then $A=(a_{ij})_{n\times n}$ has the acceptable consistency. If $GCI\ge GCI'_\alpha(n)$, then $A=(a_{ij})_{n\times n}$ has not the acceptable consistency. Here, $GCI'_\alpha(n)$ is called the threshold of GCI at the significance level α for the n –order PCM, which is denoted by

$$GCI'_\alpha(n)=\frac{2\sigma^2\chi^2_\alpha\left(\frac{n(n-1)}{2}\right)}{(n-1)(n-2)}. \tag{12}$$

Note that the critical value $\chi^2_\alpha\left(n(n-1)/2\right)$ with the degree of freedom $n(n-1)/2$ at the significance level α is obtained from Chi-squared distributions table (Johnson & Wichern, 1998), and that the variance σ^2 should be determined beforehand in order to obtain the corresponding thresholds of GCI. Taking the expectation values for both sides of (10), we obtain

$$E(\chi^2)=\frac{1}{2\sigma^2}(n-1)(n-2)E(GCI). \tag{13}$$

According to Definition 6, from (8), it follows that

$$E(\chi^2)=\frac{n(n-1)}{2}. \tag{14}$$

From (13) and (14), we obtain

$$\frac{(n-1)n}{2}=\frac{1}{2\sigma^2}(n-1)(n-2)E(GCI). \tag{15}$$

It follows that

$$\sigma^2=\frac{(n-2)}{n}E(GCI). \tag{16}$$

According to Theorem 2, we obtain

$$GCI=\frac{2n}{n-2}CI+o(\varepsilon^3). \tag{17}$$

Taking the expectation values for both sides of (17), since $E\left(o(\varepsilon^3)\right)=0$, we obtain

$$E(GCI)=\frac{2n}{n-2}E(CI). \tag{18}$$

From (16) and (18), we obtain

$$\sigma^2 = 2E(CI). \tag{19}$$

Note that $E(CI)$ is the expectation of *CI* for the PCM constructed on 1-9 scale. Moreover, *RI* is the average value of *CIs* of randomly generated positive reciprocal matrices constructed on 1–9 scale. The corresponding values of *RI* have been calculated by Saaty (1980) and Aguarón and Moreno-Jiménez (2003). It may be seen that *CI* of the PCM provided by decision maker is certainly greater than or equal to 0, and almost less than *CIs* of randomly generated positive reciprocal matrices. Accordingly, the expectation $E(CI)$ is less than the average value *RI* of *CIs*. Thus, it holds that $0 \le E(CI) < RI$. Thus, we estimate $E(CI)$ according to the relationship between $E(CI)$ and *RI*. For simplicity and convenience, the relationship between $E(CI)$ and *RI* is assumed as follows:

$$E(CI) = kRI(n)(0 \le k \le 1), \tag{20}$$

where *k* is decision coefficient denoting the assessment level of decision maker, which is assigned the value in [0, 1] according to the actual situation. If *k* = 0, then $E(CI) = 0$. That is, the assessment level of decision maker is consistent. If *k* = 1, then $E(CI) = RI$. That is, the assessment level of decision maker degenerates into random consistency level without any rational analysis.

According to (19) and (20), the variance σ^2 is estimated by

$$\hat{\sigma}^2 = 2kRI(n)(0 \le k \le 1). \tag{21}$$

From (21), $\hat{\sigma}^2$ is related to the order of the PCM and assessment level of decision maker. (Here, the difficulty in determining the variance is avoided by means of transforming it into the assessment level of decision maker.) From (12) and (21), the thresholds of GCI are denoted by

$$GCI'_{\alpha}(n) = \frac{2kRI(n)\chi^2_{\alpha}(n(n-1)/2)}{(n-1)(n-2)}. \tag{22}$$

From (22), dynamic thresholds of GCI are related to the order of the PCM, significance level and assessment level of decision maker. In order to test the consistency of the PCM, a suitable significance level α may be selected according to the actual situation. Generally, the selected significance levels are 0.01, 0.05 and 0.10.

Based on the previous discussions, the main procedure for the consistency testing of the PCM by dynamic thresholds of GCI is described as follows:

Step 1. Assign the decision coefficient *k* according to the expertise and experiences of decision maker, and then calculate $\hat{\sigma}^2$ using (21);

Step 2. Select a suitable significance level α according to the actual situation, and then calculate the threshold $GCI'_{\alpha}(n)$ using (22);

Step 3. Derive the priorities $v_i\,(1 = 1,2,\cdots,n)$ from the PCM using LLSM, and then calculate *GCI* using (1);

Step 4. Judge whether $GCI \le GCI'_{\alpha}(n)$. If this condition holds, then the PCM has the acceptable consistency; otherwise, it has not the acceptable consistency.

Note that the values of *RI* were obtained through the simulation of 500 matrices (Saaty, 1977) and 100,000 matrices (Aguarón & Moreno-Jiménez, 2003) for different order PCMs. In this paper, we used the values of *RI* obtained by Aguarón and Moreno-Jiménez (2003) because it is the most complete study, which are given in Table 2.

Table 2. Values of RI for 3- to 16-order PCMs by simulation

n	3	4	5	6	7	8	9
$RI(n)$	0.525	0.882	1.115	1.252	1.341	1.404	1.452
n	10	11	12	13	14	15	16
$RI(n)$	1.484	1.513	1.535	1.555	1.570	1.583	1.595

Without any loss of generality, we assume that the decision coefficient $k = 0.2$, and then calculate dynamic thresholds of GCI by

$$GCI'_{\alpha}(n) = \frac{0.4RI(n)\chi^2_{\alpha}(n(n-1)/2)}{(n-1)(n-2)}.$$

The corresponding thresholds of GCI for 3-to 15-order PCMs at the significance levels (0.01, 0.05 and 0.10) are shown in Table 3 (we write $M = \frac{n(n-1)}{2}$).

From Table 3, it is evident that the threshold of GCI increases as the order of the PCM increases, at the same significance level, and that the threshold of GCI increases as the significance level increases for the same order PCM, which makes up for the criticism that the approximate threshold of GCI is fixed for more than four-order PCMs. It is just in accord with the fact that different-order PCMs correspond to different thresholds of GCI.

Table 3. Dynamic thresholds of GCI for 3- to 15-order PCMs at different significance levels

n	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\chi^2_{\alpha}(M)$	$GCI'_{\alpha}(n)$	$\chi^2_{\alpha}(M)$	$GCI'_{\alpha}(n)$	$\chi^2_{\alpha}(M)$	$GCI'_{\alpha}(n)$
3	0.115	0.048	0.352	0.148	0.584	0.245
4	0.872	0.205	1.635	0.385	2.204	0.518
5	2.558	0.380	3.94	0.586	4.865	0.723
6	5.229	0.524	7.261	0.727	8.547	0.856
7	8.897	0.636	11.591	0.829	13.24	0.947
8	13.565	0.726	16.928	0.905	18.939	1.013
9	19.233	0.798	23.269	0.965	25.643	1.064
10	25.901	0.854	30.612	1.010	33.35	1.100
11	32.913	0.885	38.679	1.040	41.937	1.128
12	41.575	0.928	48.025	1.072	51.649	1.153
13	51.237	0.966	58.373	1.100	62.361	1.175
14	61.901	0.997	69.722	1.123	74.074	1.193
15	73.565	1.024	82.072	1.142	86.788	1.208

3. Numerical examples and applications

In this section, we present several numerical examples and real-world decision-making problems to illustrate the performance of dynamic thresholds of GCI by the comparative analysis with approximate thresholds of CR and GCI and other decision-making methods. Dynamic thresholds of GCI at different significance levels (0.01, 0.05 and 0.10) are shown in Table 3. (The decision coefficient of decision maker is assumed k = 0.2 in the following numerical examples.) Moreover, approximate thresholds of CR and GCI are shown in Table 1.

Example 1. We test the consistency of six groups of PCMs taken from existing references by the induced dynamic thresholds of GCI, and then compare the results with that determined by approximate thresholds of GCI and CR.

(1) The corresponding results for $A_{3\times3}$ and $A'_{3\times3}$ are shown in Table 4.

$$A_{3\times3}=\begin{pmatrix}1 & 3 & 3\\ \frac{1}{3} & 1 & 3\\ \frac{1}{3} & \frac{1}{3} & 1\end{pmatrix},\ A'_{3\times3}=\begin{pmatrix}1 & 2 & 5\\ \frac{1}{2} & 1 & 2\\ \frac{1}{5} & \frac{1}{2} & 1\end{pmatrix}.$$

Table 4. The corresponding results for two 3-order PCMs

$A_{3\times3}$		$A'_{3\times3}$	
$CR=0.1304>0.05$	NO	$CR=0.0036<0.05$	YES
$GCI=0.1341<0.3147$	YES	$GCI=0.0037<0.3147$	YES
$GCI=0.1341>0.048\ (\alpha=0.01)$	NO	$GCI=0.0037<0.048\ (\alpha=0.01)$	YES
$GCI=0.1341<0.148\ (\alpha=0.05)$	YES	$GCI=0.0037<0.148\ (\alpha=0.05)$	YES
$GCI=0.1341<0.245\ (\alpha=0.10)$	YES	$GCI=0.0037<0.245\ (\alpha=0.10)$	YES

(2) The corresponding results for $A_{4\times4}$ and $A'_{4\times4}$ are shown in Table 5.

$$A_{4\times4}=\begin{pmatrix}1 & 2 & 3 & \frac{1}{2}\\ \frac{1}{2} & 1 & 2 & 4\\ \frac{1}{3} & \frac{1}{2} & 1 & 5\\ 2 & \frac{1}{4} & \frac{1}{5} & 1\end{pmatrix},\ A'_{4\times4}=\begin{pmatrix}1 & 4 & 6 & 7\\ \frac{1}{4} & 1 & 3 & 4\\ \frac{1}{6} & \frac{1}{3} & 1 & 2\\ \frac{1}{7} & \frac{1}{4} & \frac{1}{2} & 1\end{pmatrix}.$$

Table 5. The corresponding results for two 4-order PCMs

$A_{4\times4}$		$A'_{4\times4}$	
$CR = 0.5513 > 0.08$	NO	$CR = 0.0383 < 0.08$	YES
$GCI = 1.6815 > 0.3526$	NO	$GCI = 0.1349 < 0.3526$	YES
$GCI = 1.6815 > 0.205\ (\alpha = 0.01)$	NO	$GCI = 0.1349 < 0.205\ (\alpha = 0.01)$	YES
$GCI = 1.6815 > 0.385\ (\alpha = 0.05)$	NO	$GCI = 0.1349 < 0.385\ (\alpha = 0.05)$	YES
$GCI = 1.6815 > 0.518\ (\alpha = 0.10)$	NO	$GCI = 0.1349 < 0.518\ (\alpha = 0.10)$	YES

(3) The corresponding results for $A_{5\times5}$ and $A'_{5\times5}$ are shown in Table 6.

$$A_{5\times5} = \begin{pmatrix} 1 & 2 & 6 & 3 & 3 \\ \frac{1}{2} & 1 & 2 & 5 & 4 \\ \frac{1}{6} & \frac{1}{2} & 1 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{5} & 2 & 1 & 5 \\ \frac{1}{3} & \frac{1}{4} & 1 & \frac{1}{5} & 1 \end{pmatrix}, \quad A'_{5\times5} = \begin{pmatrix} 1 & 3 & 5 & 4 & 7 \\ \frac{1}{3} & 1 & 3 & 2 & 5 \\ \frac{1}{5} & \frac{1}{3} & 1 & \frac{1}{2} & 3 \\ \frac{1}{4} & \frac{1}{2} & 2 & 1 & 3 \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}.$$

Table 6. The corresponding results for two 5-order PCMs

$A_{5\times5}$		$A'_{5\times5}$	
$CR = 0.1234 > 0.10$	NO	$CR = 0.0283 < 0.10$	YES
$GCI = 0.4305 > 0.370$	NO	$GCI = 0.1042 < 0.370$	YES
$GCI = 0.4305 > 0.380\ (\alpha = 0.01)$	NO	$GCI = 0.1042 < 0.380\ (\alpha = 0.01)$	YES
$GCI = 0.4305 < 0.586\ (\alpha = 0.05)$	YES	$GCI = 0.1042 < 0.586\ (\alpha = 0.05)$	YES
$GCI = 0.4035 < 0.723\ (\alpha = 0.10)$	YES	$GCI = 0.1042 < 0.723\ (\alpha = 0.10)$	YES

(4) The corresponding results for $A_{6\times6}$ and $A'_{6\times6}$ are shown in Table 7.

$$A_{6\times6} = \begin{pmatrix} 1 & 6 & 6 & 6 & 6 & 5 \\ \frac{1}{6} & 1 & 4 & 4 & 4 & 6 \\ \frac{1}{6} & \frac{1}{4} & 1 & 2 & 4 & 4 \\ \frac{1}{6} & \frac{1}{4} & \frac{1}{2} & 1 & 2 & 2 \\ \frac{1}{6} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \quad A'_{6\times6} = \begin{pmatrix} 1 & 2 & 4 & 6 & 8 & 9 \\ \frac{1}{2} & 1 & 2 & 4 & 6 & 8 \\ \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 & 6 \\ \frac{1}{6} & \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 \\ \frac{1}{8} & \frac{1}{6} & \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ \frac{1}{9} & \frac{1}{8} & \frac{1}{6} & \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}.$$

Table 7. The corresponding results for two 6-order PCMs

$A_{6\times 6}$		$A'_{6\times 6}$	
$CR = 0.1020 > 0.10$	NO	$CR = 0.0211 < 0.10$	YES
$GCI = 0.3585 < 0.370$	YES	$GCI = 0.0784 < 0.370$	YES
$GCI = 0.3585 < 0.524\ (\alpha = 0.01)$	YES	$GCI = 0.0784 < 0.524\ (\alpha = 0.01)$	YES
$GCI = 0.3585 < 0.727\ (\alpha = 0.05)$	YES	$GCI = 0.0784 < 0.727\ (\alpha = 0.05)$	YES
$GCI = 0.3585 < 0.856\ (\alpha = 0.10)$	YES	$GCI = 0.0784 < 0.856\ (\alpha = 0.10)$	YES

(5) The corresponding results for $A_{7\times 7}$ and $A'_{7\times 7}$ are shown in Table 8.

$$A_{7\times 7} = \begin{pmatrix} 1 & 7 & 3 & 5 & 9 & 3 & 5 \\ \frac{1}{7} & 1 & 3 & 3 & 5 & 3 & 3 \\ \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{5} & \frac{1}{3} & 3 & 3 \\ \frac{1}{5} & \frac{1}{3} & 5 & 1 & 9 & 3 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{5} & 3 & \frac{1}{9} & 1 & 3 & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{3} & \frac{1}{3} & 3 & 5 & 3 & 1 \end{pmatrix}, \quad A'_{7\times 7} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{5} & 1 & \frac{1}{4} & 2 & 3 \\ 3 & 1 & \frac{1}{2} & 2 & \frac{1}{3} & 3 & 3 \\ 5 & 2 & 1 & 4 & 5 & 6 & 5 \\ 1 & \frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{4} & 1 & 2 \\ 4 & 3 & \frac{1}{5} & 4 & 1 & 3 & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 1 & \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{5} & \frac{1}{2} & 1 & 3 & 1 \end{pmatrix}.$$

Table 8. The corresponding results for two 7-order PCMs

$A_{7\times 7}$		$A'_{7\times 7}$	
$CR = 0.3061 > 0.10$	NO	$CR = 0.1088 > 0.10$	NO
$GCI = 1.0040 > 0.370$	NO	$GCI = 0.3814 > 0.370$	NO
$GCI = 1.0040 > 0.636\ (\alpha = 0.01)$	NO	$GCI = 0.3814 < 0.636\ (\alpha = 0.01)$	YES
$GCI = 1.0040 > 0.829\ (\alpha = 0.05)$	NO	$GCI = 0.3814 < 0.829\ (\alpha = 0.05)$	YES
$GCI = 1.0040 > 0.974\ (\alpha = 0.10)$	NO	$GCI = 0.3814 < 0.947\ (\alpha = 0.10)$	YES

(6) The corresponding results for $A_{8\times 8}$ and $A'_{8\times 8}$ are shown in Table 9.

$$A_{8\times 8} = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & 1 & \frac{1}{3} & 5 & 3 & 3 & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{3} & 3 & 1 & 6 & 3 & 4 & 6 & \frac{1}{5} \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{6} & 1 & \frac{1}{3} & \frac{1}{4} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 3 & 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & 4 & 2 & 1 & \frac{1}{5} & \frac{1}{6} \\ 3 & 5 & \frac{1}{6} & 7 & 5 & 5 & 1 & \frac{1}{2} \\ 4 & 7 & 5 & 8 & 6 & 6 & 2 & 1 \end{pmatrix}, \quad A'_{8\times 8} = \begin{pmatrix} 1 & 2 & \frac{1}{2} & 2 & \frac{1}{2} & 2 & \frac{1}{2} & 2 \\ \frac{1}{2} & 1 & 4 & 1 & \frac{1}{4} & 1 & \frac{1}{4} & 1 \\ 2 & \frac{1}{4} & 1 & 4 & 1 & 4 & 1 & 4 \\ \frac{1}{2} & 1 & \frac{1}{4} & 1 & \frac{1}{4} & 1 & \frac{1}{4} & 1 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ \frac{1}{2} & 1 & \frac{1}{4} & 1 & \frac{1}{4} & 1 & \frac{1}{4} & 1 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ \frac{1}{2} & 1 & \frac{1}{4} & 1 & \frac{1}{4} & 1 & \frac{1}{4} & 1 \end{pmatrix}.$$

Table 9. The corresponding results for two 8-order PCMs

$A_{8\times 8}$		$A'_{8\times 8}$	
$CR = 0.1691 > 0.10$	NO	$CR = 0.1047 > 0.10$	NO
$GCI = 0.5292 > 0.370$	NO	$GCI = 0.2745 < 0.370$	YES
$GCI = 0.5292 < 0.726\ (\alpha = 0.01)$	YES	$GCI = 0.2745 < 0.726\ (\alpha = 0.01)$	YES
$GCI = 0.5292 < 0.905\ (\alpha = 0.05)$	YES	$GCI = 0.2745 < 0.905\ (\alpha = 0.05)$	YES
$GCI = 0.5292 < 1.013\ (\alpha = 0.10)$	YES	$GCI = 0.2745 < 1.013\ (\alpha = 0.10)$	YES

From the above results, it follows that

(1) For the PCMs with $CR \approx 0.10$ (e.g., $A_{3\times 3}$, $A_{6\times 6}$ and $A'_{8\times 8}$), the results determined by approximate thresholds of GCI are completely opposite with the ones determined by approximate thresholds of CR. The contradictory results cannot be clearly explained so far. It is quite difficult to adopt approximate thresholds of GCI or approximate thresholds of CR to test the consistency of the PCMs. In this case, we try to adopt dynamic thresholds of GCI to test the consistency of the PCMs (e.g., $A_{3\times 3}$, $A_{6\times 6}$ and $A'_{8\times 8}$), and then determine the clear results while avoiding the conflicting results. Moreover, for the PCM $A_{5\times 5}$ with CR = 0.1234, the results determined by dynamic thresholds of GCI at the significance levels (α = 0.05, 0.10) are different from the ones determined by approximate thresholds of CR and GCI. Here, only significance level sufficiently explains these different results.

(2) For the PCMs with $CR \ll 0.10$ (e.g., $A'_{3\times 3}$, $A'_{4\times 4}$, $A'_{5\times 5}$ and $A'_{6\times 6}$), the results determined by dynamic thresholds of GCI are the same as the ones determined by approximate thresholds of CR and GCI (have the acceptable consistency). Moreover, for the PCMs with $CR \gg 0.10$ (e.g., $A_{4\times 4}$ and $A_{7\times 7}$), the results determined by dynamic thresholds of GCI are the same as the ones determined by approximate thresholds of CR and GCI (have not the acceptable consistency). It is clearly concluded that the PCMs with too high or too low consistency are easily tested and have not strict requirements for the consistency index and its threshold.

From Example 1, dynamic thresholds of GCI may be a better choice due to different significance levels when other thresholds of consistency index do not accurately test the consistency of the PCM.

Example 2. We consider the PCM taken from the reference (Dadkhah & Zahedi, 1993), which was used by Amenta et al. (2020). The considered PCM (A) is listed below. The priority vectors and rank orders obtained by LLSM and EM from A are shown in Table 10.

$$A = \begin{pmatrix} 1 & 5 & 6 & 7 \\ \frac{1}{5} & 1 & 4 & 6 \\ \frac{1}{6} & \frac{1}{4} & 1 & 4 \\ \frac{1}{7} & \frac{1}{6} & \frac{1}{4} & 1 \end{pmatrix}, A' = \begin{pmatrix} 1 & 4.35 & 6 & 8.05 \\ 0.23 & 1 & 4 & 6 \\ \frac{1}{6} & \frac{1}{4} & 1 & 3.48 \\ 0.12 & \frac{1}{6} & 0.29 & 1 \end{pmatrix}, A'' = \begin{pmatrix} 1 & 3 & 6 & 9 \\ \frac{1}{3} & 1 & 3 & 5 \\ \frac{1}{6} & \frac{1}{3} & 1 & 2 \\ \frac{1}{9} & \frac{1}{5} & \frac{1}{2} & 1 \end{pmatrix}.$$

Table 10. The priority vectors and rank orders by LLSM and EM from three PCMs

Priorities	$V(A\|LLSM)$	$V(A'\|LLSM)$	$V(A\|EM)$	$V(A''\|EM)$
v_1	0.641(1)	0.611(1)	0.619(1)	0.599(1)
v_2	0.239(2)	0.246(2)	0.235(2)	0.250(2)
v_3	0.103(3)	0.099(3)	0.101(3)	0.096(3)
v_4	0.045(4)	0.045(4)	0.045(4)	0.054(4)

Amenta et al. (2020) indicated that A has not the acceptable consistency since $GCI(A) = 0.504 > 0.3526$, and then applied the revision procedure with the aim of improving the consistency and reducing $GCI(A)$ to a value below 0.3526. The revisions for some judgments are made by Amenta et al. (2020) to achieve the desired consistency ($GCI(A') = 0.344 < 0.3526$) by three iterations. The revised $PCM(A')$ (Amenta et al., 2020) is listed above. The priority vector and rank order obtained by LLSM from A' are shown in Table 10.

Dadkhah and Zahedi (1993) indicated that A has not the acceptable consistency according to the principal eigenvalue ($\lambda = 4.3907$) (In fact, $CR(A) = 0.146 > 0.08$), and then applied the multiple iteration algorithm for reaching the desired consistency ($CR(A'') = 0.013 < 0.08$). The revised PCM (A'') (Dadkhah & Zahedi, 1993) is listed above. The priority vector and rank order obtained by EM from A'' are shown in Table 10.

The changes in the derived priorities from the revised PCMs (A' and A'') are very small (That is, the derived priorities are close to the initial values.) while the updated rank orders are identical with the original rank order (1234). That is to say, the revisions of some judgments only for reaching the desired consistency are not necessary since they do not change the rank order.

However, approximate thresholds of GCI and CR do not explain the uniformity of the rank orders while not avoiding the unnecessary revisions of some judgments. Now, we try to adopt the induced dynamic thresholds of GCI to test the consistency of the PCM (A). From Table 3, when the significance levels α are 0.01, 0.05 and 0.10, dynamic thresholds of GCI (for n = 4) are 0.205, 0.385 and 0.518, respectively. Thus, A has the acceptable consistency when the significance level α is 0.10 according to $GCI(A) = 0.504 < 0.518$. Thus, it is not necessary to revise the PCM (A) with the single aim of reducing the value of GCI (CR) and reaching the desired consistency because of the uniformity and availability of the rank orders.

From Example 2, dynamic thresholds of GCI may avoid the unnecessary revision of some judgments of the PCM.

Example 3. We consider the real-world education evaluation problem taken from the reference (Lin et al., 2014). The teaching levels of six teachers are expressed by two PCMs provided by students and experts, which are $A_{6\times6}$ and $A'_{6\times6}$ in Example 1, respectively.

We should test whether the PCMs ($A_{6\times6}$ and $A'_{6\times6}$) have the acceptable consistency before ranking the teaching levels of six teachers according to the priority vectors derived from them.

For the PCM ($A_{6\times6}$) provided by students, from Table 7, $CR(A_{6\times6}) = 0.1020 > 0.10$ and $GCI(A_{6\times6}) = 0.3585 < 0.370$, thus it has not the acceptable consistency according to the approximate threshold (0.10) of CR, while it has the acceptable consistency according to the approximate threshold (0.370) of GCI. In this case, we could not directly decide whether $A_{6\times6}$ has the acceptable consistency because of two completely opposite results.

For the PCM ($A'_{6\times6}$) provided by experts, from Table 7, $CR\left(A'_{6\times6}\right) = 0.0211 < 0.10$ and $GCI\left(A'_{6\times6}\right) = 0.0784 < 0.370$, thus it has the acceptable consistency according to approximate thresholds (0.10 and 0.370) of CR and GCI. In this case, the results provided by experts are reliable according to the acceptable consistency of $A'_{6\times6}$.

The final purpose of this problem is that the teaching levels of six teachers are ranked according to the priority vector of the given PCMs. The corresponding principal eigenvectors and rank orders derived from of $A'_{6\times6}$ and $A_{6\times6}$ by EM and LLSM are shown in Table 11. The rank orders are identical (123456) according to the priority vectors derived from $A_{6\times6}$ and $A'_{6\times6}$, although the results of the consistency testing of them are completely opposite according to approximate thresholds of CR.

Table 11. The priority vectors and rank orders by LLSM and EM from two PCMs

Priorities	$V\left(A_{6\times6} \mid EM\right)$	$V\left(A'_{6\times6} \mid EM\right)$	$V\left(A_{6\times6} \mid LLSM\right)$	$V\left(A'_{6\times6} \mid LLSM\right)$
v_1	0.512(1)	0.434(1)	0.488(1)	0.429(1)
v_2	0.220(2)	0.262(2)	0.226(2)	0.265(2)
v_3	0.112(3)	0.148(3)	0.119(3)	0.149(3)
v_4	0.067(4)	0.082(4)	0.075(4)	0.082(4)
v_5	0.050(5)	0.046(5)	0.053(5)	0.046(5)
v_6	0.040(6)	0.029(6)	0.040(6)	0.028(6)

The PCM ($A'_{6\times6}$) with the acceptable consistency may guarantee that the rank order (123456) is suitable. Thus, the rank order obtained from $A_{6\times6}$ is reliable although it has not the acceptable consistency according to the approximate threshold (0.10) of CR. Moreover, the results of the consistency testing of $A_{6\times6}$ and $A'_{6\times6}$ are same according to approximate thresholds of GCI. However, approximate thresholds of GCI cannot explain conflicting results. Thus, approximate thresholds of CR and GCI, to some extent, have drawbacks for the consistency testing of the PCM.

We adopt the induced dynamic thresholds of GCI to test the consistency of the PCMs ($A_{6\times6}$ and $A'_{6\times6}$). From Table 7, $GCI\left(A_{6\times6}\right) = 0.3585$ and $GCI\left(A'_{6\times6}\right) = 0.0784$, which are less than any of the corresponding dynamic thresholds of GCI when the significance levels α are 0.01, 0.05 and 0.10 (See Table 3). Therefore, $A_{6\times6}$ and $A'_{6\times6}$ have the acceptable consistency under the selected significance levels. From Table 11, the priority vectors and rank orders obtained from $A'_{6\times6}$ and $A_{6\times6}$ by LLSM are two identical (123456). Thus, the rank orders of teaching levels of six teachers obtained from $A_{6\times6}$ and $A'_{6\times6}$ are reliable because of the acceptable consistency.

From Example 3, dynamic thresholds of GCI may explain conflicting results determined by approximate thresholds of CR and GCI for the PCM ($A_{6\times6}$), and avoid the unnecessary revisions of some judgments of the PCM for reaching the desired consistency according to the approximate threshold (0.10) of CR.

Example 4. We consider the real-world decision-making problem in water supply for the minimization of water loss by means of suitable leakage control, which is taken from the reference (Benítez et al., 2011). Two management alternatives for leakage control (Farley & Trow, 2005) are active leakage control (ALC) (which involves taking actions in distribution systems to identify and repair not reported leaks) and passive leakage control (PLC) (which boils down to just repairing reported or evident leaks).

We briefly review decision making process implemented by Benítez et al. (2011). First, the following criteria are considered for two alternatives (ALC and PLC).

C1: Planning development cost and its implementation;

C2: Damage to properties and other service networks;

C3: Effects (cost or compensations) of supply disruptions;

C4: Inconveniences caused by closed or restricted streets;

C5: Water extractions (benefits for aquifers, wetlands or rivers);

C6: Construction of tanks and reservoirs (environmental and recreational impacts);

C7: CO2 emissions.

The PCM about the relative importance among the seven criteria is $A'_{7\times 7}$ in Example 1, reflecting the opinions of a panel of experts of a water company in Valencia (Spain). Benítez et al. (2011) noted that the PCM ($A'_{7\times 7}$) is inconsistent, and then obtained the new consistent matrix ($M_{7\times 7}$) closest to it by linearization approach. The priority vector derived from $M_{7\times 7}$ by EM is

$$V\left(M_{7\times 7}\right)=\left(0.070, 0.134, 0.457, 0.089, 0.149, 0.041, 0.060\right).$$

Next, Benítez et al. (2011) constructed seven matrices of alternative comparisons according to the seven criteria, and then obtained the corresponding priority vectors of two alternatives (ALC and PLC) for each criterion. Seven priority vectors are listed as follows:

$$\begin{pmatrix} \text{Criteria} & C1 & \text{C2} & \text{C3} & \text{C4} & \text{C5} & \text{C6} & \text{C7} \\ ALC & 0.11 & 0.83 & 0.83 & 0.25 & 0.80 & 0.17 & 0.86 \\ PLC & 0.89 & 0.17 & 0.17 & 0.75 & 0.20 & 0.83 & 0.14 \end{pmatrix}.$$

These priority vectors reflect the weight or relative importance of each alternative for each criterion. Note that these priority vectors are calculated directly since any positive, reciprocal 2-order matrix is always consistent.

Finally, Benítez et al. (2011) computed the priority vector for two alternatives (ALC and PLC) by multiplying its priority value by the priority of any criterion and summing through all the criteria, which is as follows:

$$\begin{pmatrix} 0.11 & 0.83 & 0.83 & 0.25 & 0.80 & 0.17 & 0.86 \\ 0.89 & 0.17 & 0.17 & 0.75 & 0.20 & 0.83 & 0.14 \end{pmatrix} V(M_{7\times 7})^T = \begin{pmatrix} 0.695 \\ 0.305 \end{pmatrix}.$$

Benítez et al. (2011) noted that if the consistency of the PCM ($A'_{7\times 7}$) had been considered acceptable, then the priority vector derived from it would have been

$$V\left(A'_{7\times 7}\right)=\left(0.082, 0.147, 0.381, 0.072, 0.196, 0.046, 0.076\right).$$

Thus, the priority vector for two alternatives (ALC and PLC) would have been

$$\begin{pmatrix} 0.11 & 0.83 & 0.83 & 0.25 & 0.80 & 0.17 & 0.86 \\ 0.89 & 0.17 & 0.17 & 0.75 & 0.20 & 0.83 & 0.14 \end{pmatrix} V(A'_{7\times 7})^T = \begin{pmatrix} 0.698 \\ 0.302 \end{pmatrix}.$$

It is discovered that the above priority vectors for two alternatives (ALC and PLC) are almost indifferent. However, the priority vector $V\left(A'_{7\times7}\right)$ cannot be directly used to compute the priority vector for two alternatives (ALC and PLC) since the PCM ($A'_{7\times7}$) has not acceptable consistency according to the approximate threshold (0.10) of CR.

It is known that the PCM ($A'_{7\times7}$) should has the acceptable consistency before taking actions in distribution systems or individual district metered areas. Now, we adopt the induced dynamic thresholds of GCI to test the consistency of the PCM ($A'_{7\times7}$). From Table 8, $GCI\left(A'_{7\times7}\right)=0.3814$, which is less than any of the corresponding dynamic thresholds of GCI when the significance levels α are 0.01, 0.05 and 0.10 (See Table 3). That is to say, the PCM ($A'_{7\times7}$) has the acceptable consistency according to the induced dynamic thresholds of GCI. Therefore, we directly compute the priority vector for two alternatives (ALC and PLC) without revising the PCM ($A'_{7\times7}$). The priority vector derived from the PCM ($A'_{7\times7}$) by LSSM is

$$V'\left(A'_{7\times7}\right)=\left(0.081, 0.154, 0.390, 0.076, 0.182, 0.049, 0.069\right).$$

The priority vector for two alternatives (ALC and PLC) is

$$\begin{pmatrix} 0.11 & 0.83 & 0.83 & 0.25 & 0.80 & 0.17 & 0.86 \\ 0.89 & 0.17 & 0.17 & 0.75 & 0.20 & 0.83 & 0.14 \end{pmatrix} V'(A'_{7\times7})^T = \begin{pmatrix} 0.692 \\ 0.308 \end{pmatrix}.$$

This priority vector is reliable since the PCM ($A'_{7\times7}$) has the acceptable consistency according to dynamic thresholds of GCI, which is almost equivalent to the priority vector obtained by Benítez et al. (2011) using linearization approach.

From Example 4, dynamic thresholds of GCI may avoid the unnecessary revisions of the PCM by linearization approach due to the equivalent results that the priority vectors for two alternatives (ALC and PLC) are almost indifferent.

From the above examples, dynamic thresholds of GCI may clearly explain the conflicting results determined by approximate thresholds of CR and GCI, and different results at different significance levels. Sometimes, dynamic thresholds of GCI may avoid the unnecessary revisions of some judgments of the PCM for reaching the acceptable consistency according to approximate thresholds of CR and GCI. As emphasized by Dadkhah and Zahedi (1993), attaining consistency is not a goal since the inconsistency may be in the nature of the decision maker's preference.

Conclusions

AHP has been used successfully in many institutions and companies to solve the real-world decision-making problems considering all quantitative and qualitative effective factors. This paper induces dynamic thresholds of GCI associated with PCM in AHP by combining hypothesis testing and RI, which are related to the order of the PCM, significance level and assessment level of decision maker. The induced dynamic thresholds of GCI may explain different (or conflicting) results obtained by other consistency thresholds, and avoid the unnecessary revisions of some judgments of the PCM for the acceptable consistency. Compared with approximate thresholds of GCI and CR by several numerical examples and real-world decision-making problems, dynamic thresholds of GCI are more reliable and interpretable than approximate thresholds of GCI and CR. In a word, dynamic thresholds of GCI play an important role in the technological and economic sustainable development, and are available in the field of management science and operations research, including objectives, criteria, and alternatives. However, dynamic thresholds of GCI demand in application that the PCM is complete positive reciprocal matrix constructed on 1-9 scale, and cannot be directly extended to the incomplete PCM and other preference relations such as fuzzy preference, fuzzy linguistic preference, etc. In future research, we will extend dynamic thresholds of GCI to the incomplete PCM and other preference relations for sustainability performance measurement and assessment.

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References

Aguarón, J., & Moreno-Jiménez, J. M. (2003). The geometric consistency index: approximate thresholds. *European Journal of Operational Research*, *147*(1), 137–145. https://doi.org/10.1016/S0377-2217(02)00255-2

Aguarón, J., Escobar, M. T., & Moreno-Jiménez., J. M. (2021). Reducing inconsistency measured by the geometric consistency index in the analytic hierarchy process. *European Journal of Operational Research*, *288*(2), 576–583. https://doi.org/10.1016/j.ejor.2020.06.014

Altuzarra, A., Moreno-Jiménez, J. M., & Salvador, M. (2007). A Bayesian priorization procedure for AHP-group decision making. *European Journal of Operational Research*, *182*(1), 367–382. https://doi.org/10.1016/j.ejor.2006.07.025

Amenta, P., Lucadamo, A., & Marcarelli, G. (2018). Approximate thresholds for Salo-Hamalainen index. *IFAC*, *51*(11), 1655–1659. https://doi.org/10.1016/j.ifacol.2018.08.219

Amenta, P., Lucadamo, A., & Marcarelli, G. (2020). On the transitivity and consistency approximate thresholds of some consistency indices for pairwise comparison matrices. *Information Sciences*, *507*, 274–287. https://doi.org/10.1016/j.ins.2019.08.042

Banae, C., & Vansnick, J. (2008). A critical analysis of the eigenvalue method used to derive priorities in AHP. *European Journal of Operational Research*, *187*(3), 1422–1428. https://doi.org/10.1016/j.ejor.2006.09.022

Barzilai, J. (1997). Deriving weights from pairwise comparison matrices. *Journal of the Operational Research Society*, *48*(12), 1226–1232. https://doi.org/10.2307/3010752

Barzilai, J., & Golany, B. (1994). AHP rank reversal, normalization and aggregation rules. *INFOR: Information Systems and Operational Research*, *32*(2), 57–63. https://doi.org/10.1080/03155986.1994.11732238

Barzilai, J., Cook, W. D., & Golany, B. (1987). Consistent weights for judgments matrices of the relative importance of alternatives. *Operations Research Letters*, *6*(3), 131–134. https://doi.org/10.1016/0167-6377(87)90026-5

Benítez, J., Delgado-Galván, X., Izquierdo, J., & Pérez-García, R. (2011). Achieving matrix consistency in AHP through linearization. *Applied Mathematical Modelling*, *35*(9), 4449–4457. https://doi.org/10.1016/j.apm.2011.03.013

Bernardo, J. M., & Smith, A. F. M. (1994). *Bayesian theory*. Wiley. https://doi.org/10.1002/9780470316870

Bozóki S., & Rapcsák, T. (2008). On *Saaty's* and *Koczkodaj's* inconsistencies of pairwise comparison matrices. *Journal of Global Optimization*, *42*(2), 157–175. https://doi.org/10.1007/s10898-007-9236-z

Bozóki, S., Dezső, L., Poesz, A., & Temesi, J. (2013). Analysis of pairwise comparison matrices: an empirical research. *Annals of Operations Research*, *211*(1), 511–528. https://doi.org/10.1007/s10479-013-1328-1

Brugha, C. (1996). The structure of qualitative decision-making: Implications for the analytical hierarchy process. In *The International Symposium on the Analytic Hierarchy Process* (pp. 190–201). https://doi.org/10.13033/isahp.y1996.076

Brugha, C. M. (2004). Structure of multi-criteria decision-making. *Journal of the Operational Research Society*, *55*(11), 1156–1168. https://doi.org/10.1057/palgrave.jors.2601777

Brunelli, M. (2016). A technical note on two inconsistency indices for preference relations: A case of functional relation. *Information Sciences*, *357*, 1–5. https://doi.org/10.1016/j.ins.2016.03.048

Brunelli, M. (2017). Studying a set of properties of inconsistency indices for pairwise comparisons. *Annals of Operations Research*, *248*(1), 143–161. https://doi.org/10.1007/s10479-016-2166-8

Brunelli, M. (2018). A survey of inconsistency indices for pairwise comparisons. *International Journal of General Systems*, *47*(8), 751–771. https://doi.org/10.1080/03081079.2018.1523156

Brunelli, M., & Fedrizzi, M. (2015). Axiomatic properties of inconsistency indices for pairwise comparisons. *Journal of the Operational Research Society*, *66*(1), 1–15. https://doi.org/10.1057/jors.2013.135

Cavallo, B. (2020). Functional relations and Spearman correlation between consistency indices. *Journal of the Operational Research Society*, *71*(2), 301–311. https://doi.org/10.1080/01605682.2018.1516178

Cavallo, B., & D'Apuzzo, L. (2009). A general unified framework for pairwise comparison matrices in multicriterial methods. *International Journal of Intelligent Systems*, *24*(4), 377–398. https://doi.org/10.1002/int.20329

Cavallo, B., & D'Apuzzo, L. (2010). Characterizations of consistent pairwise comparison matrices over abelian linearly ordered groups. *International Journal of Intelligent Systems*, *25*(10), 1035–1059. https://doi.org/10.1002/int.20438

Crawford, G., & Williams, C. (1985). A note on the analysis of subjective judgment matrices. *Journal of Mathematical Psychology*, *29*(4), 387–405. https://doi.org/10.1016/0022-2496(85)90002-1

Csató, L. (2018a). Characterization of an inconsistency ranking for pairwise comparison matrices. *Annals of Operations Research*, *261*, 155–165. https://doi.org/10.1007/s10479-017-2627-8

Csató, L. (2018b). Characterization of the row geometric mean ranking with a group consensus axiom. *Group Decision and Negotiation*, *27*, 1011–1027. https://doi.org/10.1007/s10726-018-9589-3

Csató, L. (2019a). Axiomatizations of inconsistency indices for triads. *Annals of Operations Research*, *280*(1–2), 99–110. https://doi.org/10.1007/s10479-019-03312-0

Csató, L. (2019b). A characterization of the Logarithmic Least Squares Method. *European Journal of Operational Research*, *276*(1), 212–216. https://doi.org/10.1016/j.ejor.2018.12.046

Csató, L., & Petróczy, D. G. (2020). On the monotonicity of the eigenvector method. *European Journal of Operational Research*, *292*(1), 230–237. https://doi.org/10.1016/j.ejor.2020.10.020

Dadkhah, K. M., & Zahedi, F. (1993). A mathematical treatment of inconsistency in the analytic hierarchy process. *Mathematical and Computer Modelling*, *17*(4), 111–122. https://doi.org/10.1016/0895-7177(93)90180-7

Devore, T. L. (2000). *Probability and statistics*. Thomson Learning.

Dixit, P. D. (2018). Entropy production rate as a criterion for inconsistency in decision theory. *Journal of Statistical Mechanics: Theory and Experiment*, *2018*(5), 053408. https://doi.org/10.1088/1742-5468/aac137

Duszak, Z., & Koczkodaj, W. (1994). Generalization of a new definition of consistency for pairwise comparisons. *Information Processing Letters*, *52*(5), 273–276. https://doi.org/10.1016/0020-0190(94)00155-3

Escobar, M. T., & Moreno-Jiménez, J. M. (2000). Reciprocal distributions in the analytic hierarchy process. *European Journal of Operational Research*, *123*(1), 154–174. https://doi.org/10.1016/S0377-2217(99)00086-7

Farley, M., & Trow, S. (2005). *Losses in water distribution networks. A practitioner's guide to assessment, monitoring and control*. IWA Publishing. https://doi.org/10.2166/9781780402642

Fedrizzi, M., & Ferrari, F. (2018). A chi-square-based inconsistency index for pairwise comparison matrices. *Journal of the Operational Research Society*, *69*(7), 1125–1134. https://doi.org/10.1080/01605682.2017.1390523

Fedrizzi, M., & Giove, S. (2007). Incomplete pairwise comparisons and consistency optimization. *European Journal of Operational Research*, *183*(1), 303–313. https://doi.org/10.1016/j.ejor.2006.09.065

Gass, S. I., & Rapcsák, T. (2004). Singular value decomposition in AHP. *European Journal of Operational Research*, *154*(3), 573–584. https://doi.org/10.1016/S0377-2217(02)00755-5

Georgiou, D., Mohammed, E. S., & Rozakis, S. (2015). Multi-criteria decision making on the energy supply configuration of autonomous desalination units. *Renewable Energy*, *75*, 459–467. https://doi.org/10.1016/j.renene.2014.09.036

Grzybowski, A. Z. (2016). New results on inconsistency indices and their relationship with the quality of priority vector estimation. *Expert Systems with Applications*, *48*(C), 130. https://doi.org/10.1016/j.eswa.2015.08.049

Hamchaoui, S., Boudoukha, A., & Benzerra, A. (2015). Drinking water supply service management and sustainable development challenges: Case study of Bejaia, Algeria. *Journal of Water Supply: Research and Technology-Aqua*, *64*(8), 937–946. https://doi.org/10.2166/aqua.2015.156

Ishizaka, A., & Labib, A. (2011). Review of the main developments in the analytic hierarchy process. *Expert Systems with Applications*, *38*(11), 14336–14345. https://doi.org/10.1016/j.eswa.2011.04.143

Jin, F., Ni, Z., Chen, H., & Li, Y. (2016). Approaches to decision making with linguistic preference relations based on additive consistency. *Applied Soft Computing*, *49*, 71–80. https://doi.org/10.1016/j.asoc.2016.07.045

Jin, F., Ni, Z., Langari, R., & Chen, H. (2020). Consistency improvement-driven decision-making methods with probabilistic multiplicative preference relations. *Group Decision and Negotiation*, *29*, 371–397. https://doi.org/10.1007/s10726-020-09658-2

Jin, F., Liu, J., Zhou, L., & Martínez, L. (2021). Consensus-based linguistic distribution large-scale group decision making using statistical inference and regret theory. *Group Decision and Negotiation*, *30*, 813–845. https://doi.org/10.1007/s10726-021-09736-z

Johnson, R. A., & Wichern, D. W. (1998). *Applied multivariate statistical analysis*. Prentice Hall.

Karapetrovic, S., & Rosenbloom, E. A. (1999). Quality control approach to consistency paradoxes in AHP. *European Journal of Operational Research*, *119*(3), 704–718. https://doi.org/10.1016/S0377-2217(98)00334-8

Koczkodaj, W. W. (1993). A new definition of consistency for pairwise comparisons. *Mathematical and Computer Modelling*, *18*(7), 79–84. https://doi.org/10.1016/0895-7177(93)90059-8

Koczkodaj, W., & Szwarc, R. (2014). On Axiomatization of inconsistency indicators for pairwise comparisons. *Fundamenta Informaticae*, *132*(4), 485–500. https://www.deepdyve.com/lp/ios-press/on-axiomatization-of-inconsistency-indicators-for-pairwise-comparisons-KvdxIkyigt

Koczkodaj, W. W., & Urban, R. (2018). Axiomatization of inconsistency indicators for pairwise comparisons. *International Journal of Approximate Reasoning*, *94*, 18–29. https://doi.org/10.1016/j.ijar.2017.12.001

Kou, G., & Lin, C. (2014). A cosine maximization method for the priority vector derivation in AHP. *European Journal of Operational Research*, *235*(1), 225–232. https://doi.org/10.1016/j.ejor.2013.10.019

Kou, G., Ergu, D., Chen, Y., & Lin, C. (2016). Pairwise comparison matrix in multiple criteria decision making. *Technological and Economic Development of Economy*, *22*(5), 738–765. https://doi.org/10.3846/20294913.2016.1210694

Kou, G., Peng, Y., & Wang, G. (2014). Evaluation of clustering algorithms for financial risk analysis using MCDM methods. *Information Sciences*, *275*(11), 1–12. https://doi.org/10.1016/j.ins.2014.02.137

Kou, G., Yang, P., Peng, Y., Xiao, F., Chen, Y., & Alsaadi, F. E. (2020). Evaluation of feature selection methods for text classification with small datasets using multiple criteria decision-making methods. *Applied Soft Computing Journal*, *86*, 105836. https://doi.org/10.1016/j.asoc.2019.105836

Kułakowski, K. (2015). Notes on order preservation and consistency in AHP. *European Journal of Operational Research*, *245*(1), 333–337. https://doi.org/10.1016/j.ejor.2015.03.010

Kwiesielewicz, M., & Uden, E. (2004). Inconsistent and contradictory judgements in pairwise comparison method in AHP. *Computers & Operations Research*, *31*(5), 713–719. https://doi.org/10.1016/S0305-0548(03)00022-4

Lin, C., Kou, G., & Ergu, D. (2013). A heuristic approach for deriving the priority vector in AHP. *Applied Mathematical Modelling*, *37*(8), 5828–5836. https://doi.org/10.1016/j.apm.2012.11.023

Lin, C., Kou, G., & Ergu, D. (2014). A statistical approach to measure the consistency level of the pairwise comparison matrix. *Journal of the Operational Research Society*, *65*(9), 1380–1386. https://doi.org/10.1057/jors.2013.92

Lundy, M., Siraj, S., & Greco, S. (2017). The mathematical equivalence of the "spanning tree" and row geometric mean preference vectors and its implications for preference analysis. *European Journal of Operational Research*, *257*(1), 197–208. https://doi.org/10.1016/j.ejor.2016.07.042

Marques, R. C., Cruz, N. F., & Pires, J. (2015). Measuring the sustainability of urban water services. *Environmental Science & Policy*, *54*, 142–151. https://doi.org/10.1016/j.envsci.2015.07.003

Monsuur, H. (1997). An intrinsic consistency threshold for reciprocal matrices. *European Journal of Operational Research*, *96*(2), 387–391. https://doi.org/10.1016/S0377-2217(96)00372-4

Peláez, J., & Lamata, M. (2003). A new measure of consistency for positive reciprocal matrices. *Computer & Mathematics with Applications*, *46*(12), 1839–1845. https://doi.org/10.1016/S0898-1221(03)90240-9

Saaty, T. L. (1977). A scaling method for priorities in hierarchical structures. *Journal of Mathematical Psychology*, *15*(3), 234–281. https://doi.org/10.1016/0022-2496(77)90033-5

Saaty, T. L. (1980). *The analytic hierarchy process*. McGraw-Hill.

Saaty, T. L. (1994). *Fundamentals of decision making and priority theory with the analytic hierarchy process*. RSW Publication.

Saaty, T. L. (2013). The modern science of multicriteria decision making and its practical applications: *The AHP/ANP Approach. Operations Research*, *61*(5), 1101–1118. https://doi.org/10.1287/opre.2013.1197

Salo, A., & Hamalainen, R. (1997). On the measurement of preference in the analytic hierarchy process. *Journal of Multi-Criteria Decision Analysis*, *6*(6), 309–319. https://doi.org/10.1002/(SICI)1099-1360(199711)6:6<309::AID-MCDA163>3.0.CO;2-2

Siraj, S., Mikhailov, L., & Keane, J. (2012). A heuristic method to rectify intransitive judgments in pairwise comparison matrices. *European Journal of Operational Research*, *216*(2), 420–428. https://doi.org/10.1016/j.ejor.2011.07.034

Siraj, S., Mikhailov, L., & Keane, J. (2015). Contribution of individual judgments toward inconsistency in pairwise comparisons. *European Journal of Operational Research*, *242*(2), 557–567. https://doi.org/10.1016/j.ejor.2014.10.024

Stein, W. E., & Mizzi, P. J. (2007). The harmonic consistency index for the analytic hierarchy process. *European Journal of Operational Research*, *177*(1), 488–497. https://doi.org/10.1016/j.ejor.2005.10.057

Vargas, L. G. (1982). Reciprocal matrices with random coefficients. *Mathematical Modelling*, *3*(1), 69–81. https://doi.org/10.1016/0270-0255(82)90013-6